

On Metric Dimension of Functigraphs

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Abstract

The *metric dimension* of a graph G , denoted by $\dim(G)$, is the minimum number of vertices such that each vertex is uniquely determined by its distances to the chosen vertices. Let G_1 and G_2 be disjoint copies of a graph G and let $f : V(G_1) \rightarrow V(G_2)$ be a function. Then a *functigraph* $C(G, f) = (V, E)$ has the vertex set $V = V(G_1) \cup V(G_2)$ and the edge set $E = E(G_1) \cup E(G_2) \cup \{uv \mid v = f(u)\}$. We study how metric dimension behaves in passing from G to $C(G, f)$ by first showing that $2 \leq \dim(C(G, f)) \leq 2n - 3$, if G is a connected graph of order $n \geq 3$ and f is any function. We further investigate the metric dimension of functigraphs on complete graphs and on cycles.

Key Words: distance, resolving set, metric dimension, functigraph, complete graph, cycle

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1 Introduction

Let $G = (V(G), E(G))$ be a simple, undirected, connected, and nontrivial graph with order $|V(G)|$. The *degree* of a vertex v in G , denoted by $\deg_G(v)$, is the number of edges that are incident to v in G ; an *end-vertex* is a vertex of degree one, and a *support vertex* is a vertex that is adjacent to an end-vertex. For a vertex $v \in V(G)$, the *open neighborhood* of v is the set $N_G(v) = \{u \mid uv \in E(G)\}$, and the *closed neighborhood* of v is the set $N_G[v] = N_G(v) \cup \{v\}$. For $S \subseteq V(G)$, the *open neighborhood* of S is the set $N_G(S) = \cup_{v \in S} N_G(v)$ and the *closed neighborhood* of S is the set $N_G[S] = N_G(S) \cup S$; throughout the paper, we denote by $N(S)$ ($N[S]$, respectively) the open (closed, respectively) neighborhood of S in $C(G, f)$. We denote by K_n , C_n , and P_n the complete graph, the cycle, and the path on n vertices, respectively. The *distance* between two vertices $v, w \in V(G)$ is denoted by $d_G(v, w)$; throughout the paper, we denote by $d(v, w)$ the distance between v and w in $C(G, f)$. For an ordered set $S = \{u_1, u_2, \dots, u_k\} \subseteq V(G)$ of distinct vertices, the *metric code* (or *code*, in short) of $v \in V(G)$ with respect to S is the k -vector $\text{code}_S(v) = (d_G(v, u_1), d_G(v, u_2), \dots, d_G(v, u_k))$. A vertex $x \in V(G)$ *resolves* a pair of vertices $v, w \in V(G)$ if $d_G(v, x) \neq d_G(w, x)$. A set of vertices $S \subseteq V(G)$ *resolves* G if every pair of distinct vertices of G are resolved by some vertex in S or, equivalently, if $\text{code}_S(u) \neq \text{code}_S(v)$ for distinct vertices u and v of G ; then S is called a *resolving set* of G . The *metric dimension* of G , denoted by $\dim(G)$, is the minimum of $|S|$ as S varies over all resolving sets of G . For other terminologies in graph theory, we refer to [8].

Slater [23, 24] introduced the concept of a resolving set for a connected graph under the term *locating set*; he referred to a minimum resolving set as a *reference set*, and the cardinality of a minimum

resolving set as the *location number* of a graph. Independently, Harary and Melter [16] studied these concepts under the term *metric dimension*. Metric dimension as a graph parameter has numerous applications, among them are robot navigation [19], sonar [23], combinatorial optimization [21], and pharmaceutical chemistry [5]. It was noted in [14] that determining the metric dimension of a graph is an NP-hard problem. Metric dimension has been heavily studied; for surveys, see [1] and [9]. For more articles on metric dimension in graphs, see [2], [4], [5], [7], [12], [13], [15], [19], [20], and [22].

Chartrand and Harary [6] introduced a “permutation graph” (or “generalized prism”). Hedetniemi [17] introduced a “function graph”, which comprises two graphs (not necessarily identical copies) with a function relation between them. Independently, Dörfler [11] introduced a “mapping graph”, which consists of two disjoint identical copies of a graph and additional edges between the two vertex sets specified by a function. The “mapping graph” was rediscovered and studied in [10], where it was called a “functigraph”. We recall the definition of the functigraph.

Definition 1.1. Let G_1 and G_2 be disjoint copies of a graph G , and let $f : V(G_1) \rightarrow V(G_2)$ be a function. A *functigraph* $C(G, f) = (V, E)$ consists of the vertex set $V = V(G_1) \cup V(G_2)$ and the edge set $E(G) = E(G_1) \cup E(G_2) \cup \{uv \mid v = f(u)\}$.

In this paper, we study the metric dimension of functigraphs. For a connected graph G of order $n \geq 3$ and for a function f , we show that $2 \leq \dim(C(G, f)) \leq 2n - 3$. We provide in Remark 2.6 an example showing that $\dim(G) - \dim(C(G, f))$ can be arbitrarily large; we also show in Theorem 4.5 that $\dim(C(C_n, f))$ can be arbitrarily large for a constant function f , though $\dim(C_n) = 2$. These examples are quite surprising; they indicate the complexity, vis-à-vis metric dimension, present in passing from G to $C(G, f)$. Further, we give the metric dimension of functigraphs on complete graphs and we also give bounds for the metric dimension of functigraphs on cycles. It is worth noting that the metric dimension of the wheel graph, which is a subgraph of $C(C_n, f)$ for a constant function f , has been studied in [3] and [22].

2 Bounds on Metric Dimension of Functigraphs

We first recall some basic facts on metric dimension for background.

Theorem 2.1. [5] For a connected graph G of order $n \geq 2$ and diameter d ,

$$f(n, d) \leq \dim(G) \leq n - d,$$

where $f(n, d)$ is the least positive integer k for which $k + d^k \geq n$.

A generalization of Theorem 2.1 has been given in [18] by Hernando et al.

Theorem 2.2. [18] Let G be a graph of order n , diameter $d \geq 2$, and metric dimension k . Then

$$n \leq \left(\left\lfloor \frac{2d}{3} \right\rfloor + 1 \right)^k + k \sum_{i=1}^{\lceil \frac{d}{3} \rceil} (2i - 1)^{k-1}.$$

Theorem 2.3. [5] Let G be a connected graph of order $n \geq 2$. Then

- (a) $\dim(G) = 1$ if and only if $G = P_n$,
- (b) $\dim(G) = n - 1$ if and only if $G = K_n$,
- (c) for $n \geq 4$, $\dim(G) = n - 2$ if and only if $G = K_{s,t}$ ($s, t \geq 1$), $G = K_s + \overline{K}_t$ ($s \geq 1, t \geq 2$), or $G = K_s + (K_1 \cup K_t)$ ($s, t \geq 1$); here, $A + B$ denotes the graph obtained from the disjoint union of graphs A and B by joining every vertex of A with every vertex of B , and \overline{C} denotes the complement of a graph C .

Next, we obtain general bounds for the metric dimension of functigraphs. If G is a connected graph of order 2, then $G \cong P_2$ and $\dim(C(P_2, f)) = 2$ for any function f . So, we only consider a connected graph G of order $n \geq 3$ for the rest of the paper.

Theorem 2.4. *Let G be a connected graph of order $n \geq 3$, and let $f : V(G_1) \rightarrow V(G_2)$ be a function. Then $2 \leq \dim(C(G, f)) \leq 2n - 3$. Both bounds are sharp.*

Proof. Since $C(G, f)$ contains a cycle, $\dim(C(G, f)) \geq 2$ by (a) of Theorem 2.3. On the other hand, noting that $C(G, f) \not\cong K_{2n}$ for any function f , $\dim(C(G, f)) \leq 2n - 2$ by (b) of Theorem 2.3; further, no $C(G, f)$ satisfies $\dim(C(G, f)) = 2n - 2$ by (c) of Theorem 2.3, and thus $\dim(C(G, f)) \leq 2n - 3$. For the sharpness of the lower bound, take $G = P_n$ and $f \equiv id$, the identity function; then $\dim(C(P_n, id)) = 2$, since two end-vertices of $G_1 \cong P_n$ form a minimum resolving set for $C(P_n, id)$. For the sharpness of the upper bound, $G = K_n$, with f a constant function, is an example – this will be shown in Theorem 3.4. \square

The following definitions are stated in [5]. Fix a graph G . A vertex of degree at least three is called a *major vertex*. An end-vertex u is called a *terminal vertex* of a major vertex v if $d_G(u, v) < d_G(u, w)$ for every other major vertex w . The *terminal degree* of a major vertex v is the number of terminal vertices of v . A major vertex v is an *exterior major vertex* if it has positive terminal degree. Let $\sigma(G)$ denote the sum of terminal degrees of all major vertices of G , and let $ex(G)$ denote the number of exterior major vertices of G . Next, we recall the concept of *twin vertices*. Two vertices $u, v \in V(G)$ are called *twins* if $N_G(u) - \{v\} = N_G(v) - \{u\}$; notice that for any set S with $S \cap \{u, v\} = \emptyset$, $code_S(u) = code_S(v)$.

Theorem 2.5. [5, 19, 20] *If T is a tree that is not a path, then $\dim(T) = \sigma(T) - ex(T)$.*

Remark 2.6. *The graph in Figure 1 shows that $\dim(G) - \dim(C(G, f))$ can be arbitrarily large. Let f_i denote the restriction of f on $V(G_1^i)$, where $1 \leq i \leq n$. Notice $\dim(G_1^i) = \sigma(G_1^i) - ex(G_1^i) = 6 - 1 = 5$. But $\dim(C(G_1^i, f_i)) \leq 4$, since the solid vertices in $C(G_1^i, f_i)$ form a resolving set for $C(G_1^i, f_i)$, as one can explicitly check. Also, notice that, for any $u, v \in V(C(G_1^i, f_i))$, the distance between u and v in $C(G_1, f)$ is not less than their distance in $C(G_1^i, f_i)$; i.e., the solid vertices of $C(G_1^i, f_i)$ still distinguish the vertices of $C(G_1^i, f_i)$ in $C(G_1, f)$. Thus, $\dim(C(G_1, f)) \leq n \cdot \dim(C(G_1^i, f_i)) \leq 4n$. But, $\dim(G_1) = n \cdot \dim(G_1^i) = 5n$. Therefore, $\dim(G_1) - \dim(C(G_1, f)) \geq n$.*

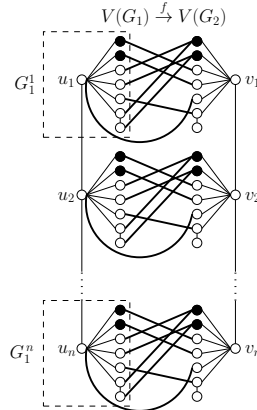


Figure 1: An example showing that $\dim(G) - \dim(C(G, f))$ can be arbitrarily large

3 Metric Dimension of Functigraphs on Complete Graphs

In this section, for $n \geq 3$, we show that 1) $\dim(C(K_n, \sigma)) = n - 1$ for a permutation σ ; 2) $\dim(C(K_n, f)) = 2n - 2 - |f(V(G_1))|$ for a non-permutation function f . Throughout this section, we let $V(G_1) = \{u_i \mid 1 \leq i \leq n\}$ and $V(G_2) = \{v_i \mid 1 \leq i \leq n\}$ for $G_1 \cong G_2 \cong K_n$, where $n \geq 3$. Noting that each vertex in a resolving set S has a distinct code from any other vertex in $C(G, f)$, one only needs to check the codes of vertices in $V(C(G, f)) - S$.

We recall the following theorem, which can be viewed as a functigraph where the function is the identity on a connected graph G .

Theorem 3.1. [5] *For every connected graph G , $\dim(G) \leq \dim(G \square K_2) \leq \dim(G) + 1$, where $A \square B$ denotes the Cartesian product of two graphs A and B .*

Theorem 3.2. [4] *For every graph G and for all $n \geq 2 \dim(G) + 1$, $\dim(K_n \square G) = n - 1$.*

As an immediate corollary of Theorem 3.2, we have the following

Corollary 3.3. *Let $G = K_n$ be the complete graph of order $n \geq 3$, and let $\sigma : V(G_1) \rightarrow V(G_2)$ be a permutation. Then $\dim(C(K_n, \sigma)) = n - 1$.*

Theorem 3.4. *Let $G = K_n$ be the complete graph of order $n \geq 3$, and let $f_0 : V(G_1) \rightarrow V(G_2)$ be a constant function. Then $\dim(C(K_n, f_0)) = 2n - 3$.*

Proof. Without loss of generality, let $f_0(u_i) = v_1$ for each i ($1 \leq i \leq n$). Let S be a minimum resolving set of $C(K_n, f_0)$. First, we will show that $|S| \geq 2n - 3$ for $n \geq 3$. Since any two vertices in G_1 are twins, all but one of these n vertices must be in S . Similarly, since any two vertices in $\{v_2, v_3, \dots, v_n\}$ are twins, all but one of these $(n-1)$ vertices must be in S . So, $|S| \geq (n-1) + (n-2) = 2n - 3$. By Theorem 2.4, $|S| \leq 2n - 3$, and thus $\dim(C(K_n, f_0)) = 2n - 3$. More explicitly, one can check that $S = \{u_1, u_2, \dots, u_{n-1}, v_2, v_3, \dots, v_{n-1}\}$ is a resolving set for $C(K_n, f_0)$. \square

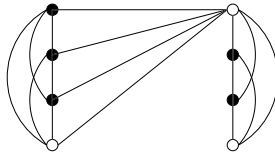


Figure 2: $\dim(C(K_4, f_0)) = 5$ for a constant function f_0

Remark 3.5. *Theorem 3.4 may be generalized as follows. For $m, n \geq 3$, let $H_1 = K_m$ and $H_2 = K_n$. Let $f_0 : V(H_1) \rightarrow V(H_2)$ be a constant function. Let $\mathcal{G} = (V, E)$ be the graph with $V = V(H_1) \cup V(H_2)$ and $E = E(H_1) \cup E(H_2) \cup \{u_i v_1 \mid f_0(u_i) = v_1 \text{ for each } i (1 \leq i \leq m)\}$. Then $\dim(\mathcal{G}) = m + n - 3$.*

Theorem 3.6. *Let $G = K_n$ be the complete graph of order $n \geq 3$, and let $|f(V(G_1))| = s$ where $1 < s < n$. Then $\dim(C(K_n, f)) = 2n - 2 - s$.*

Proof. Without loss of generality, we may assume that $f(V(G_1)) = \{v_1, v_2, \dots, v_s\}$ such that $|f^{-1}(v_i)| = k_i$ with $k_1 \geq k_2 \geq \dots \geq k_s \geq 1$ and $\sum_{i=1}^s k_i = n$. Further, we may assume that $f^{-1}(v_1) = \{u_i \mid 1 \leq i \leq k_1\}$, $f^{-1}(v_2) = \{u_i \mid k_1 + 1 \leq i \leq k_1 + k_2\}$, \dots , and $f^{-1}(v_s) = \{u_i \mid 1 + \sum_{t=1}^{s-1} k_t \leq i \leq \sum_{t=1}^s k_t\}$; we adopt the convention that $\sum_{i=a}^b f(i) = 0$ when $b < a$. Since $|f(V(G_1))| < n$, $k_1 \geq 2$.

First, we show that $\dim(C(K_n, f)) \geq 2n - 2 - s$. Let S be any minimum resolving set of $C(K_n, f)$. Since any two vertices in $\{v_{s+1}, v_{s+2}, \dots, v_n\}$ are twins, all but one of these $(n - s)$ vertices must be in S , say $S_0 = \{v_{s+1}, v_{s+2}, \dots, v_{n-1}\} \subseteq S$ with $|S_0| = n - s - 1$. Similarly, for each i ($1 \leq i \leq s$), $f^{-1}(v_i)$ consists of k_i vertices such that any two vertices in $f^{-1}(v_i)$ are twins, and thus $(k_i - 1)$ of each $f^{-1}(v_i)$ must be in S ; we may assume that $S_1 = \cup_{i=1}^s \{u_j \mid 1 + \sum_{t=1}^{i-1} k_t \leq j \leq (\sum_{t=1}^i k_t) - 1\} \subseteq S$ with $|S_1| = n - s$. We denote by $a_i = \sum_{t=1}^i k_t$ for each i ($1 \leq i \leq s$). For each i and j , $1 \leq i < j \leq s$, consider the two vertices u_{a_i} and u_{a_j} . They are both distance 1 from every u_ℓ with $\ell \neq a_i$ and $\ell \neq a_j$ and distance 2 from every v_m with $m \neq i$ and $m \neq j$. Thus, to resolve these two vertices, one of the vertices in $\{u_{a_i}, u_{a_j}, v_i, v_j\}$ must be in S . Since this is true for every pair i and j with $1 \leq i < j \leq s$, there exists at most one ℓ such that $\{u_{a_\ell}, v_\ell\} \cap S = \emptyset$. Thus, we have $|S| \geq (n - s - 1) + (n - s) + (s - 1) = 2n - 2 - s$.

Next, we show that $S = \{u_2, u_3, \dots, u_n, v_{s+1}, v_{s+2}, \dots, v_{n-1}\}$ is a resolving set of $C(K_n, f)$ with $|S| = (n - 1) + (n - 1 - s) = 2n - 2 - s$. Clearly, (i) u_1 has 1 in the k -th entry and 2 in the rest of the entries of its code, where $1 \leq k \leq n - 1$; (ii) for $1 \leq i \leq s$, each v_i has 1 in the k -th entry and 2 in the rest of the entries of its code, where $n \leq k \leq 2n - 2 - s$ or $\max\{1, \sum_{t=1}^{i-1} k_t\} \leq k \leq (\sum_{t=1}^i k_t) - 1$; (iii) v_n has 2 in the k -th entry and 1 in the rest of the entries of its code, where $1 \leq k \leq n - 1$. Thus, S is a resolving set for $C(K_n, f)$, and hence $\dim(C(K_n, f)) \leq 2n - 2 - s$ for $1 < s < n$.

Therefore, $\dim(C(K_n, f)) = 2n - 2 - s$ for $1 < s < n$. \square

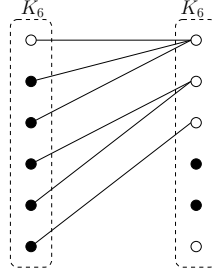


Figure 3: $\dim(C(K_6, f)) = 7$, where $|f(V(G_1))| = 3$

4 Metric Dimension of Functigraphs on Cycles

In this section, we give bounds of the metric dimension of functigraphs on cycles. Let $G = C_n$ for $n \geq 3$, and let G_1 and G_2 be disjoint copies of G . Let $V(G_1) = \{u_i \mid 1 \leq i \leq n\}$ and let $E(G_1) = \{u_i u_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{u_1 u_n\}$; similarly, let $V(G_2) = \{v_i \mid 1 \leq i \leq n\}$ and let $E(G_2) = \{v_i v_{i+1} \mid 1 \leq i \leq n - 1\} \cup \{v_1 v_n\}$.

Proposition 4.1. [4] For $n \geq 3$,

$$\dim(C_n \square K_2) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even} . \end{cases}$$

Theorem 4.2. Let $G = C_n$ be the cycle of order $n \geq 3$, and let $\sigma : V(G_1) \rightarrow V(G_2)$ be a permutation. Then $2 \leq \dim(C(C_n, \sigma)) \leq n - 1$, and both bounds are attainable.

Proof. By Theorem 2.4, $\dim(C(C_n, \sigma)) \geq 2$. Next, we show that $\dim(C(C_n, \sigma)) \leq n - 1$. We will show that $S = \{u_2, u_3, \dots, u_n\}$ is a resolving set of $C(C_n, \sigma)$. Note that (i) u_1 has 1 exactly in the 1st and in the $(n - 1)$ th entries of its code; (ii) $\sigma(u_1)$ does not contain 1 in any entry of its code;

(iii) for $2 \leq i \leq n$, each $\sigma(u_i)$ has 1 exactly in the $(i-1)$ th entry of its code. Since S is a resolving set of $C(C_n, \sigma)$ with $|S| = n-1$, $\dim(C(C_n, \sigma)) \leq n-1$. For the sharpness of the lower bound, $\dim(C(C_5, id)) = 2$ by Proposition 4.1; for the sharpness of the upper bound, $\dim(C(C_4, id)) = 3$ by Proposition 4.1. \square

We recall the following

Theorem 4.3. [3, 22] For $n \geq 3$, let $W_{1,n} = C_n + K_1$ be the wheel graph on $n+1$ vertices. Then

$$\dim(W_{1,n}) = \begin{cases} 3 & \text{if } n = 3 \text{ or } n = 6, \\ \lfloor \frac{2n+2}{5} \rfloor & \text{otherwise.} \end{cases}$$

In order to determine the metric dimension of $C(C_n, f_0)$ for a constant function f_0 , we need the following lemma.

Lemma 4.4. Let $G = C_n$ be the cycle of order $n \geq 6$, and let $f_0 : V(G_1) \rightarrow V(G_2)$ satisfy $f_0(u_i) = v_1$ for each i ($1 \leq i \leq n$). Let S be a minimum resolving set of $C(C_n, f_0)$.

- (a) There exists at most one vertex u in G_1 such that $u \notin N[S]$. Moreover, if C_n is an even cycle and $|V(G_1) - N[S]| = 1$, then at least two vertices in G_2 must belong to S .
- (b) Let $S \cap V(G_1) = \{u_{i_1}, u_{i_2}, \dots, u_{i_t}\}$, where $i_1 < i_2 < \dots < i_t$. If $i_{j+1} - i_j = 3$ for some j , say $2 \leq j \leq n-2$ by relabeling if necessary, then $d_{G_1}(u_{i_j}, u_{i_{j-1}}) \leq 2$ and $d_{G_1}(u_{i_{j+2}}, u_{i_{j+1}}) \leq 2$.
- (c) For each $u_i \in S \cap V(G_1)$, to resolve $A = N(u_i) \cap V(G_1)$, at least a vertex in $(N[A] - \{u_i\}) \cap V(G_1)$ must belong to S .

Proof. (a) Suppose there exist at least two vertices in $V(G_1) - N[S]$, say $u_p, u_q \in V(G_1) - N[S]$. Then, for each $x \in S \cap V(G_1)$, $d(u_p, x) = d(u_q, x) = 2$. Noting that no vertex in G_2 resolves any two vertices in G_1 , we have $code_S(u_p) = code_S(u_q)$. Thus at most one vertex belongs to $V(G_1) - N[S]$. Next, suppose that $u \in V(G_1) - N[S]$ and C_n is an even cycle. Notice that, for each $x \in S \cap V(G_1)$, $d(u, x) = d(v_2, x) = d(v_n, x) = 2$. Assume that $|S \cap V(G_2)| = 1$. Since neither v_1 nor $v_{\frac{n}{2}+1}$ resolves v_2 and v_n , we may assume that $y \in S \cap V(G_2)$ for some $y \in \{v_2, v_3, \dots, v_{\frac{n}{2}}\}$. But, $d(y, u) = d(y, v_1) + 1 = d(y, v_n)$ for $y \in \{v_2, v_3, \dots, v_{\frac{n}{2}}\}$. Thus, $|S \cap V(G_2)| \geq 2$ in this case.

(b) Suppose that $i_{j+1} - i_j = 3$ for some j with $2 \leq j \leq n-2$ (by relabeling if necessary), and that $d_{G_1}(u_{i_j}, u_{i_{j-1}}) \geq 3$ or $d_{G_1}(u_{i_{j+2}}, u_{i_{j+1}}) \geq 3$. Without loss of generality, assume that $d_{G_1}(u_{i_j}, u_{i_{j-1}}) \geq 3$. Then the two vertices in $N(u_{i_j}) \cap V(G_1)$ have the same code.

(c) Let $u_i \in S \cap V(G_1)$, where $3 \leq i \leq n-2$ by relabeling if necessary. If $\{u_{i-2}, u_{i-1}, u_{i+1}, u_{i+2}\} \cap S = \emptyset$, then $code_S(u_{i-1}) = code_S(u_{i+1})$. \square

Theorem 4.5. Let $G = C_n$ be the cycle of order $n \geq 3$, and let $f_0 : V(G_1) \rightarrow V(G_2)$ be a constant function. Then

$$\dim(C(C_n, f_0)) = \begin{cases} 3 & \text{if } n = 3, \\ \lfloor \frac{2n+3}{5} \rfloor & \text{if } n \text{ is odd and } n \neq 3, \\ \lfloor \frac{2n}{5} \rfloor + 1 & \text{if } n \text{ is even.} \end{cases}$$

Proof. Without loss of generality, let $f_0(u_i) = v_1$ for each i ($1 \leq i \leq n$). Let S be a minimum resolving set of $C(C_n, f_0)$, where $n \geq 3$. First, we consider $3 \leq n \leq 5$. Notice that no vertex of G_2 resolves any two vertices of G_1 , so at least two vertices of G_1 must be in S . Furthermore, at least a vertex in $V(G_2) - \{v_1\}$ must belong to S , since v_2 and v_n are not resolved by any vertex of G_1 . So, $|S| \geq 3$ for $3 \leq n \leq 5$. One easily checks that $\{u_1, u_3, v_3\}$ is a resolving set for $C(C_3, f_0)$ and $C(C_5, f_0)$. It's also easily checked that $\{u_1, u_2, v_2\}$ is a resolving set for $C(C_4, f_0)$. Thus, $\dim(C(C_n, f_0)) = 3$, consistent with the formula asserted in our theorem, for $3 \leq n \leq 5$.

Next, we consider for $n \geq 6$. Notice that no vertex of G_2 resolves any two vertices of G_1 , and, as already observed, at least a vertex of $V(G_2) - \{v_1\}$ must belong to S .

Claim 1. For $n \geq 6$, $|S| \geq \lceil \frac{2n+3}{5} \rceil$ if n is odd and $|S| \geq \lceil \frac{2n}{5} \rceil + 1$ if n is even.

Proof of Claim 1. Let $S \cap V(G_1) = \{u_{i_1}, u_{i_2}, \dots, u_{i_t}\}$, where $i_1 < i_2 < \dots < i_t$, with $|S \cap V(G_1)| = t$. We consider two cases.

Case 1. There is a vertex $u \in V(G_1)$ such that $u \notin N[S] \cap V(G_1)$: Without loss of generality, we may assume that $\{u_1, u_2, \dots, u_{n-1}\} \subseteq N[S] \cap V(G_1)$ and $u_n \notin N[S] \cap V(G_1)$. Then $i_1 = 2$, $i_t = n - 2$, and, by (b) of Lemma 4.4, we have $i_2 - i_1 \leq 2$ and $i_t - i_{t-1} \leq 2$. By Lemma 4.4, $n - t \leq 3(\frac{n-6}{5}) + 4 = \frac{3n+2}{5}$ (see (A) of Figure 4); thus $t \geq \frac{2n-2}{5}$. If n is odd, noting that $|S \cap V(G_2)| \geq 1$, $|S| \geq \lceil \frac{2n-2}{5} \rceil + 1 = \lceil \frac{2n+3}{5} \rceil$. If n is even, by (a) of Lemma 4.4, $|S| \geq \lceil \frac{2n+3}{5} \rceil + 1$.

Case 2. There is no vertex $u \in V(G_1)$ such that $u \notin N[S] \cap V(G_1)$: In this case, $\{u_1, u_2, \dots, u_n\} \subseteq N[S] \cap V(G_1)$. By Lemma 4.4, $n - t \leq 3(\frac{n-5}{5}) + 3 = \frac{3n}{5}$ (see (B) of Figure 4); thus $t \geq \frac{2n}{5}$. Since $|S \cap V(G_2)| \geq 1$, we have $|S| \geq \lceil \frac{2n}{5} \rceil + 1$.

Since one of these two cases must occur, for n odd, $\dim(C(C_n, f_0)) \geq \min\{\lceil \frac{2n+3}{5} \rceil, \lceil \frac{2n+5}{5} \rceil\} = \lceil \frac{2n+3}{5} \rceil$. For n even, $\dim(C(C_n, f_0)) \geq \min\{\lceil \frac{2n+3}{5} \rceil + 1, \lceil \frac{2n+5}{5} \rceil\} = \lceil \frac{2n+5}{5} \rceil$. \square

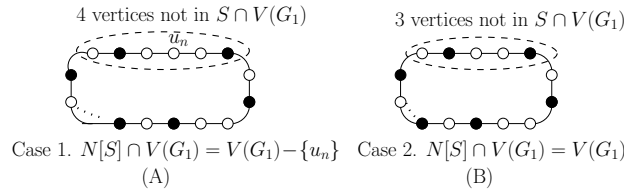


Figure 4: The solid vertices in diagram (A) (resp., (B)) form $S \cap V(G_1)$ in Case 1 (resp., Case 2).

Claim 2. For $n \geq 6$, $|S| \leq \lceil \frac{2n+3}{5} \rceil$ if n is odd and $|S| \leq \lceil \frac{2n}{5} \rceil + 1$ if n is even.

Proof of Claim 2. We will show the existence of a resolving set S of $C(C_n, f_0)$ of cardinalities given by the bounds. We consider two cases.

Case 1. $n \equiv 0, 2, 4 \pmod{5}$: If $n \equiv 0 \pmod{5}$, then $S = \{u_{5i+2}, u_{5i+5} \mid 0 \leq i \leq \frac{n-5}{5}\} \cup \{v_2\}$ is a resolving set of $C(C_n, f_0)$; if $n \equiv 2 \pmod{5}$, then $S = \{u_{5i+2}, u_{5i+5} \mid 0 \leq i \leq \frac{n-7}{5}\} \cup \{u_n\} \cup \{v_2\}$ is a resolving set of $C(C_n, f_0)$; if $n \equiv 4 \pmod{5}$, then $S = \{u_{5i+2}, u_{5i+5} \mid 0 \leq i \leq \frac{n-9}{5}\} \cup \{u_{n-2}, u_n\} \cup \{v_2\}$ is a resolving set of $C(C_n, f_0)$. Since $N[S] \cap V(G_1) = V(G_1)$ and $|S \cap V(G_1)| \geq 3$ for $n \geq 6$, by (b) and (c) of Lemma 4.4, $S \cap V(G_1)$ resolves all vertices in G_1 and no vertex in G_1 has the same code with a vertex in G_2 . Further, a vertex in $S \cap V(G_1)$ and v_2 resolves all vertices in G_2 . Thus, S is a resolving set of $C(C_n, f_0)$ with $|S| = \lceil \frac{2n}{5} \rceil + 1 = \lceil \frac{2n+3}{5} \rceil$.

Case 2. $n \equiv 1, 3 \pmod{5}$: If n is odd and $n \equiv 1 \pmod{5}$, then $S = \{u_2, u_{n-2}\} \cup \{u_{5i+4}, u_{5i+7} \mid 0 \leq i \leq \frac{n-11}{5}\} \cup \{v_{\frac{n+1}{2}}\}$ is a resolving set of $C(C_n, f_0)$ with $|S| = \frac{2n+3}{5} = \lceil \frac{2n+3}{5} \rceil$ (see (A) of Figure 5); if n is odd and $n \equiv 3 \pmod{5}$, then $S = \{u_2, u_{n-4}, u_{n-2}\} \cup \{u_{5i+4}, u_{5i+7} \mid 0 \leq i \leq \frac{n-13}{5}\} \cup \{v_{\frac{n+1}{2}}\}$ is a resolving set of $C(C_n, f_0)$ with $|S| = \frac{2n+4}{5} = \lceil \frac{2n+3}{5} \rceil$. If n is even and $n \equiv 1 \pmod{5}$, then $S = \{u_2, u_{n-2}\} \cup \{u_{5i+4}, u_{5i+7} \mid 0 \leq i \leq \frac{n-11}{5}\} \cup \{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\}$ is a resolving set of $C(C_n, f_0)$ with $|S| = \frac{2n+3}{5} + 1 = \frac{2n+8}{5} = \lceil \frac{2n}{5} \rceil + 1$ (see (B) of Figure 5); if n is even and $n \equiv 3 \pmod{5}$, then $S = \{u_2, u_{n-4}, u_{n-2}\} \cup \{u_{5i+4}, u_{5i+7} \mid 0 \leq i \leq \frac{n-13}{5}\} \cup \{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\}$ is a resolving set of $C(C_n, f_0)$ with $|S| = \frac{2n+4}{5} + 1 = \frac{2n+9}{5} = \lceil \frac{2n}{5} \rceil + 1$. For each case, noting that $\{u_n\} = V(G_1) - N[S]$, by (b) and (c) of Lemma 4.4, $S_1 = S \cap V(G_1)$ resolves all vertices but u_n in G_1 and $\text{code}_{S_1}(u_n) = \text{code}_{S_1}(v_2) =$

$code_{S_1}(v_n)$. If n is odd, a vertex in $S \cap V(G_1)$ and $v_{\frac{n+1}{2}}$ resolve all vertices in G_2 ; further, noting that $d(u_n, v_{\frac{n+1}{2}}) = 1 + d(v_1, v_{\frac{n+1}{2}})$, we have $d(u_n, v_{\frac{n+1}{2}}) > \max\{d(v_2, v_{\frac{n+1}{2}}), d(v_n, v_{\frac{n+1}{2}})\}$, and hence $code_S(u_n) \neq code_S(v_2)$ and $code_S(u_n) \neq code_S(v_n)$. If n is even, a vertex in $S \cap V(G_1)$ and $\{v_{\frac{n}{2}}, v_{\frac{n}{2}+1}\}$ resolve all vertices in G_2 ; further, noting that $d(u_n, v_{\frac{n}{2}}) = 1 + d(v_1, v_{\frac{n}{2}}) > d(v_2, v_{\frac{n}{2}})$ and $d(u_n, v_{\frac{n}{2}+1}) = 1 + d(v_1, v_{\frac{n}{2}+1}) > d(v_n, v_{\frac{n}{2}+1})$, we have $code_S(u_n) \neq code_S(v_2)$ and $code_S(u_n) \neq code_S(v_n)$. Thus, S is a resolving set of $C(C_n, f_0)$.

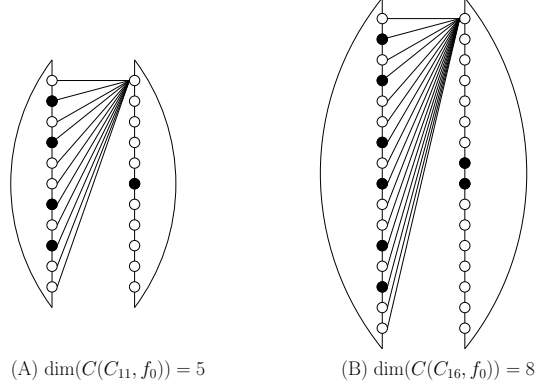


Figure 5: The metric dimension of $C(C_{11}, f_0)$ and $C(C_{16}, f_0)$ and their minimum resolving sets, where the black vertices in each functigraph form a minimum resolving set

Therefore, for $n \geq 6$, by Claim 1 and Claim 2, we have $\dim(C(C_n, f_0)) = \lceil \frac{2n+3}{5} \rceil$ if n is odd and $\dim(C(C_n, f_0)) = \lceil \frac{2n}{5} \rceil + 1$ if n is even. \square

Theorem 4.6. For $n \geq 3$, let $G = C_n$. Let $f : V(G_1) \rightarrow V(G_2)$ be any function with $|f(V(G_1))| = s$, where $1 < s < n$. Then $2 \leq \dim(C(C_n, f)) \leq 2(n-1) - s$.

Proof. By Theorem 2.4, $\dim(C(C_n, f)) \geq 2$. Let $W = f(V(G_1))$. We will show that $\dim(C(C_n, f)) \leq 2(n-1) - |W|$. We consider two cases.

Case 1. $|W| = n-1$: Without loss of generality, we may assume that $|f^{-1}(v_1)| = 2$, say $f^{-1}(v_1) = \{u_1, u_x\}$, by relabeling if necessary. Let $v_y \in V(G_2) - W$. We will show that $S = \{u_2, u_3, \dots, u_n\}$ is a resolving set of $C(C_n, f)$. Note that (i) u_1 has 1 exactly in the 1st and in the $(n-1)$ th entries of its code; (ii) $f(u_1)$ has 1 exactly in the $(x-1)$ -th entry of its code for some x ($2 \leq x \leq n$) such that $u_x \in f^{-1}(v_1)$; (iii) for $2 \leq i \leq n$, each $f(u_i)$ has 1 exactly in the $(i-1)$ th entry of its code; (iv) v_y does not contain 1 in any entry of its code. Since S is a resolving set of $C(C_n, f)$ with $|S| = n-1$, $\dim(C(C_n, f)) \leq n-1 = 2(n-1) - |W|$.

Case 2. $2 \leq |W| \leq n-2$: In this case, $n \geq 4$. For $C(C_4, f)$ with $|f(V(G_1))| = 2$, there are six non-isomorphic graphs (see Figure 6); one can readily check that $\dim(C(C_4, f)) \leq 4$ for each case.

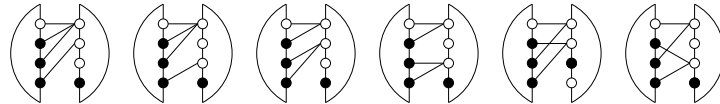


Figure 6: Six non-isomorphic $C(C_4, f)$ with $|f(V(G_1))| = 2$, where the black vertices in each functigraph form a resolving set

So, let $n \geq 5$, and we consider two subcases.

Subcase 2.1. There exists a vertex, say u_1 , in G_1 with $N(u_1) \cap V(G_1) = \{u_2, u_n\}$ such that $f(u_2) \neq f(u_n)$: Let $S_2 = V(G_2) - (W \cup \{v'\})$ for some $v' \in V(G_2) - W$; we note here that $S_2 \neq \emptyset$. Since $|V(G_2) - W| \geq 2$ and $n \geq 5$, there exists a vertex, say $v' \notin W$, in G_2 such that $N(v') \cap V(G_2) \neq \{f(u_2), f(u_n)\}$. We will show that $S = S_1 \cup S_2$ is a resolving set of $C(C_n, f)$, where $S_1 = \{u_2, u_3, \dots, u_n\}$ and $S_2 = \{v_1, v_2, \dots, v_n\} - (W \cup \{v'\})$. Note that, for the 1st through the $(n-1)$ th entries of its code, (i) u_1 has 1 exactly in the 1st and in the $(n-1)$ th entries of its code; (ii) if $|f^{-1}(f(u_1))| = 1$, then $f(u_1)$ does not contain 1 but has 2 in the 1st and in the $(n-1)$ th entries of its code; if $|f^{-1}(f(u_1))| \geq 2$, then see (iii); (iii) for $2 \leq i \leq n$, each $f(u_i)$ has 1 in the $(i-1)$ th entry of its code, but no $f(u_i)$ has 1 in the 1st and in the $(n-1)$ th entries of its code at the same time; further, there's exactly one vertex in $W \subset V(G_2)$ with 1 in the $(i-1)$ th entry of its code; (iv) $v' \in V(G_2)$ does not contain 1 in the 1st through $(n-1)$ th entries of its code, and v' does not have 2 in the 1st and in the $(n-1)$ th entries of its code at the same time either. Since S is a resolving set of $C(C_n, f)$ with $|S| = (n-1) + (n-s-1) = 2(n-1) - s$, $\dim(C(C_n, f)) \leq 2(n-1) - s$.

Subcase 2.2. For each vertex, say u , in G_1 , two vertices in $N(u) \cap V(G_1)$ are mapped to the same vertex in G_2 : Notice that $|W| = 1$ if C_n is an odd cycle, and $|W| = 2$ if C_n is an even cycle. Since $|W| \geq 2$, C_n must be an even cycle and $f(u_1) = f(u_3) = \dots = f(u_{n-1}) = v'$ and $f(u_2) = f(u_4) = \dots = f(u_n) = v''$ for $v' \neq v''$. In this case, $S = S_1 \cup S_2$ is a resolving set of $C(C_n, f)$, where $S_1 = \{u_3, u_4, \dots, u_n\}$ and $S_2 = \{v_1, v_2, \dots, v_n\} - \{v', v''\}$. Note that, for the 1st through the $(n-2)$ th entries of its code, (i) u_1 has 1 only in the $(n-2)$ th entry; (ii) u_2 has 1 only in the 1st entry; (iii) v' has 1 in the ℓ_1 -th entry, where $1 \leq \ell_1 \leq n-2$ and ℓ_1 is odd; (iv) v'' has 1 in the ℓ_2 -th entry, where $1 \leq \ell_2 \leq n-2$ and ℓ_2 is even. Since S is a resolving set of $C(C_n, f)$, $\dim(C(C_n, f)) \leq 2n-4 = 2(n-1) - |W|$. \square

Remark 4.7. For the graph in Figure 7, where $n = 3$ and $|f(V(G_1))| = 2$, the formula for the upper bound in Theorem 4.6 yields 2, which is also the lower bound. Notice that $\{u_2, u_3\}$ is a minimum resolving set of $C(C_3, f)$.

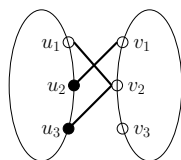


Figure 7: A functigraph showing the sharpness of the bounds in Theorem 4.6

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